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2-balanced flows and the inverse 1-median problem in the Chebyshev space

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ABSTRACT

In this paper, we consider the 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm. We give an optimality criterion for this problem which enables us to solve the following inverse location problem by a combinatorial algorithm in polynomial time: Given n points $P_1, \dots, P_n \in \mathbb{R}^d$ with non-negative weights w_i and a point P_0 the task is to find new non-negative weights $\tilde{w}_i$ such that P_0 is a 1-median with respect to the new weights and $\|w - \tilde{w}\|_1$ is minimized. In fact, this problem reduces to a 2-balanced flow problem for which an optimal solution can be obtained by solving a fractional b-matching problem.

1. Introduction

Location problems are an important field of operations research with many practical applications. The roots of location problems can be seen in an essay of Pierre de Fermat where he asked the question how a single facility should be placed in the plane such that the sum of the distances from a given set of three points to the new facility is minimized. Afterwards, this model was generalized by Weber [1] who considered n points with positive weights and the task is to find a location that minimizes the sum of the weighted distances to the given points. Nowadays, this problem is called the 1-median problem in the Euclidean plane and is also known as the Fermat–Weber problem. It is usually solved by an iterative algorithm which is based on an idea of Weiszfeld (see e.g., Drezner et al. [2] and the references therein).

Recently, a lot of other location problems have been investigated. The number of facilities that are allowed to be located was not fixed to one any more and different spaces were considered. Typically, one distinguishes between continuous location problems, discrete location problems and network location problems. In the first case facilities have to be located in some d-dimensional space like $\mathbb{R}^d$ whereas in a discrete location problem there is only a discrete set of potential locations. In network location problems the facilities can be placed on the vertices of a graph or in the interior of an edge. In addition, several different objective functions were introduced. The most common is the minimization of the sum of the weighted distances (median problems). However, center problems where one is interested in minimizing the largest weighted distance to a point are also well studied.

This paper focuses on the weighted 1-median problem in $\mathbb{R}^d$ where the distance of two points is measured by the Chebyshev-norm. This problem can be solved in linear time for $d = 2$ (see e.g., Hamacher [3]), because in this special case the Chebyshev-metric and the Manhattan-metric are related by a linear transformation. However, for $d \geq 3$ the same idea does not work any more, because the topology of these two norms is totally different. Hatzl and Karrenbauer [4] propose a first combinatorial algorithm for the d-dimensional case by reducing the problem to a min-cost-flow problem in a bipartite graph. Moreover, they prove that there is a 1-median, which is half-integral, provided that the given points have integral coordinates.

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The aim of this paper is to solve the inverse 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm. In an inverse optimization problem, a feasible solution is already given and the task is to modify some parameters of the problem at minimum cost such that the given solution is optimal with respect to the modified instance. This kind of problem was investigated for many combinatorial optimization problems, e.g., shortest path problem, maximum flow problem, spanning tree problem. For further results and solution methods on inverse optimization problems we refer to the survey paper of Heuberger [5]. In an inverse location problem the locations of the facilities are already given and one is allowed to change the edge lengths (if the problem is defined on a network) and/or the vertex weights. Depending on the model, there may also be some bounds on the parameters that are allowed to be modified. Moreover, there are different ways how to measure the total cost for changing the parameters.

Burkard et al. [6] considered the inverse p-median problem on graphs where it is allowed to change the vertex weights within some bounds. They prove that this problem can be solved in polynomial time if the asymmetric ℓ_1-norm is used as cost function. They also show that if $p = 1$ and the problem is defined on a tree the problem reduces to a continuous knapsack-problem which can be solved in linear time. Later, the same authors developed in [7] an $\mathcal{O}(n^2)$ algorithm if the underlying graph is a cycle and $p = 1$. The case where it is allowed to change the edge lengths of the graph was recently discussed by Baroughi et al. [8]. Cai et al. [9] prove that the inverse 1-center problem is strongly $\mathcal{NP}$-hard if the weights can be changed. This result is quite interesting, because the 1-center problem can be solved in polynomial time, but its inverse problem is hard. However, if one is allowed to change the edge lengths and the graph is a tree, the inverse problem is again solvable in polynomial time [10].

Inverse continuous location problems were less considered so far. Burkard et al. [11] prove that the inverse Fermat–Weber problem with variable weights and the ℓ_1-norm as cost function can be solved by a combinatorial algorithm in $\mathcal{O}(n \log n)$ if the prespecified point does not coincide with a given point. Another continuous inverse 1-median problem is discussed in [6]. Here, the authors focus on the Manhattan-metric and assume that the costs for changing the weights depend on the vertices.

In this paper, we solve the inverse 1-median problem with respect to the Chebyshev-norm. This problem was not investigated so far, because there was no optimality condition known. As a consequence it was difficult to find a combinatorial algorithm. Here we state the first combinatorial optimality criterion. Based on this result we develop an efficient algorithm for the inverse location problem.

This paper is organized as follows: In the next section the problems under consideration are defined. In Section 3, some preliminary results for the two-dimensional case are given which are helpful in order to deal with higher dimensions. Then, we state an optimality criterion for the considered 1-median problem and show that the corresponding inverse problem can be described as a 2-balanced flow problem. In Section 5, we discuss how the 2-balanced flow problem (and therefore also the inverse 1-median problem) can be transformed to a fractional b-matching problem which can be solved in polynomial time.

2. Problem formulation

In this section, we formally define the 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm and the corresponding inverse problem. Let us start with the classical location problem: Given n points $P_1, \ldots, P_n$ with $P_i = (x_1^i, \ldots, x_d^i) \in \mathbb{R}^d$ for $i = 1, \ldots, n$ and associated non-negative weights $w_i \geq 0$ the task is to find a point $P^* = (x_1^*, \ldots, x_d^*) \in \mathbb{R}^d$ such that

$$\sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty \geq \sum_{i=1}^{n} w_i \, \|P_i - P^*\|_\infty$$

holds for all $P \in \mathbb{R}^d$, where $\|P_i - P^*\|_\infty := \max (|x_1^i - x_1^*|, \ldots, |x_d^i - x_d^*|)$ is the Chebyshev-norm. Such a point P^* is called 1-median.

Note that the problem of finding the 1-median with respect to the Chebyshev-norm in the d-dimensional space can be written as a linear programming problem. To see this let us introduce new variables z_i which give the distance from P_i to the 1-median and rewrite the problem

$$\min_{P=(y_1,\ldots,y_d)\in\mathbb{R}^d} \sum_{i=1}^{n} w_i \, \|P_i - P\|_\infty$$

as

$$\begin{aligned} &\min \sum_{i=1}^{n} w_i \, z_i \\ &\text{s.t.} \quad z_i = \max (|x_1^i - y_1|, \ldots, |x_d^i - y_d|) \quad i = 1, \ldots, n \\ &\qquad\ \ y_j \in \mathbb{R}, \; z_i \in \mathbb{R} \quad j = 1, \ldots, d, \; i = 1, \ldots, n. \end{aligned} \tag{1}$$

It is a well known fact that constraints of the form given in (1) can be transformed to an LP in standard form which has the following form in our case:

$$\min \sum_{i=1}^{n} w_i z_i \tag{2}$$

$$\text{s.t.} \quad z_i + y_j \geq x_j^i \quad i = 1, \ldots, n,\ j = 1, \ldots, d \tag{3}$$

$$z_i - y_j \geq -x_j^i \quad i = 1, \ldots, n,\ j = 1, \ldots, d \tag{4}$$

$$y_j \in \mathbb{R},\ z_i \in \mathbb{R} \quad j = 1, \ldots, d,\ i = 1, \ldots, n. \tag{5}$$

Due the fact that the problem can be formulated as linear programming problem, the problem can be solved in polynomial time. A combinatorial algorithm for this problem can be found in [4].

In this paper, we are mainly interested in the inverse location problem. While the task of the 1-median problem is to find a point that minimizes the sum of the weighted distances, in the inverse problem a solution of the location problem is already given. Here, the goal is to change the weights of the vertices at minimum cost such that the prespecified point becomes an optimal solution. In this paper, the costs for changing the weights are given by the ℓ_1-norm. Hence, an instance of the inverse problem is given by a set of n points $P_1, \ldots, P_n$ with corresponding non-negative weights $w_i \geq 0$ and a point P_0 (which may coincide with a given point). The task is to find new weights $\tilde{w}_i \geq 0$ such that P_0 is a 1-median with respect to the weighs $\tilde{w}$ and $\|\tilde{w} - w\|_1$ is minimized. Concluding, the problem can be formulated in a compact form as follows:

$$\begin{aligned} \min\ & \sum_{i=1}^{n} |w_i - \tilde{w}_i| \\ \text{s.t.}\ & \sum_{i=1}^{n} \tilde{w}_i \|P_i - P\|_\infty \geq \sum_{i=1}^{n} \tilde{w}_i \|P_i - P_0\|_\infty \quad \forall P \in \mathbb{R}^d \\ & 0 \leq \tilde{w}_i \quad i = 1, \ldots, n. \end{aligned}$$

Note that this problem has an infinite number of constraints, so that it cannot be concluded immediately that the inverse problem is solvable in polynomial time. However, the task of the inverse location problem is in fact to change the cost coefficients of the linear programming problem (2)–(5) at minimum cost such that a given solution becomes optimal. Thus, the problem can be seen as a special case of an inverse linear programming problem that was investigated in Ahuja and Orlin [12]. They show that the inverse linear programming problem is again a linear programming problem and can thus be solved in polynomial time. As a consequence the inverse location problem defined above is solvable in polynomial time. In the rest of the paper, we follow the ideas of [12] but develop a combinatorial algorithm by considering some features that arise for this special problem.

Finally, we want to point out that the problem discussed in [11] is very similar to the problem investigated here. Therein, the authors give a linear time algorithm for the problem

$$\begin{aligned} \min\ & \sum_{i=1}^{n} |w_i - \tilde{w}_i| \\ \text{s.t.}\ & \sum_{i=1}^{n} \tilde{w}_i \|P_i - P\|_2 \geq \sum_{i=1}^{n} \tilde{w}_i \|P_i - P_0\|_2 \quad \forall P \in \mathbb{R}^2 \\ & \underline{w}_i \leq w_i - \tilde{w}_i \leq \overline{w}_i \quad i = 1, \ldots, n \end{aligned}$$

with given bounds $\underline{w}_i \leq 0 \leq \overline{w}_i$. In fact, the problem above is the inverse 1-median problem in $\mathbb{R}^2$ with respect to the Euclidean distance. So the main difference is the different norm in the underlying location problem. This fact asks for new ideas and the results and the algorithm from [11] cannot be used here.

3. The two-dimensional case

Before we discuss an optimality criterion for the 1-median problem introduced in the previous section we state a simple proposition which will make notation much easier in the remaining part of the paper.

Proposition 3.1. *Assume we are given n points $P_1, \ldots, P_n$ in $\mathbb{R}^d$ with associated weights $w_i \geq 0$. Then, P^* is a 1-median if and only if the origin $P_0 = (0, \ldots, 0)$ is a 1-median for the points $Q_i := P_i - P^*$ with weights w_i.*

This proposition follows immediately from the fact that the Chebyshev-norm is invariant with respect to translation. Using Proposition 3.1 it suffices to give an optimality criterion for the origin. Furthermore, we assume without loss of generality that the origin is the point that is given in the inverse location problem and has to be optimal with respect to the modified vertex weights.

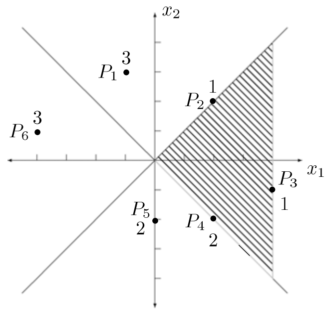

Fig. 1. The instance of Example 3.3 where the cone $Q_1^{\geq}$ is highlighted.

To give an optimality criterion for two dimensions we use the notation

$$W(X_1 \sim X_2) := \sum_{\{i: x_1^i \sim x_2^i\}} w_i \quad \text{and} \quad W(X_1 \sim X_2^-) := \sum_{\{i: x_1^i \sim -x_2^i\}} w_i$$

where $\sim$ may be any of the relations $<, \leq, >, \geq$. Then, the following result for two dimensions is well known.

Theorem 3.2 (*See e.g. [6,3]*). *The point $P^* = (0, 0)$ is an optimal solution of the 1-median problem in $\mathbb{R}^2$ with respect to the Chebyshev-norm if and only if the following inequalities are satisfied:*

$$W(X_1 > X_2) \leq W(X_1 \leq X_2) \tag{6}$$
$$W(X_1 < X_2) \leq W(X_1 \geq X_2) \tag{7}$$
$$W(X_1 < X_2^-) \leq W(X_1 \geq X_2^-) \tag{8}$$
$$W(X_1 > X_2^-) \leq W(X_1 \leq X_2^-). \tag{9}$$

Let us consider the following example in order to demonstrate the notation and the theorem.

Example 3.3. Suppose we are given the following points: $P_1 = (-1, 3), P_2 = (2, 2), P_3 = (4, -1), P_4 = (2, -2), P_5 = (0, -2)$ and $P_6 = (-4, 1)$ with the weights $w_1 = 3$, $w_2 = 1$, $w_3 = 1$, $w_4 = 2$, $w_5 = 2$ and $w_6 = 3$ (see Fig. 1). Then, we have

$$W(X_1 > X_2) = 5 \leq 7 = W(X_1 \leq X_2)$$
$$W(X_1 < X_2) = 6 \leq 6 = W(X_1 \geq X_2)$$
$$W(X_1 < X_2^-) = 5 \leq 7 = W(X_1 \geq X_2^-)$$
$$W(X_1 > X_2^-) = 5 \leq 7 = W(X_1 \leq X_2^-)$$

and can conclude that $P^* = (0, 0)$ is an optimal solution.

The proof of Theorem 3.2 relies mainly on the fact that for two points $P_1 = (x_1^1, x_2^1)$ and $P_2 = (x_1^2, x_2^2)$

$$\|P_1 - P_2\|_\infty = \|T(P_1) - T(P_2)\|_1 \tag{10}$$

holds in the plane, where $T(P) = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(-x_1 + x_2)\right)$ for a point $P = (x_1, x_2)$. Due to the fact that an optimality criterion for the 1-median problem in $\mathbb{R}^d$ with respect to the ℓ_1-norm is known (see e.g., [6]) the above result can easily be derived. However, the proof cannot be extended to higher dimensions, because a transformation similar to (10) does not exist for $d \geq 3$. Thus, we have to develop different ideas to obtain an optimality criterion that is also valid for $d \geq 3$.

Let us start the investigations with some more notations which will be helpful later on. We define the cones

$$Q_j^{\geq} := \{x \in \mathbb{R}^d : x_j \geq 0 \text{ and } |x_j| > |x_k| \; \forall k \neq j\}$$

and

$$Q_j^{\leq} := \{x \in \mathbb{R}^d : x_j \leq 0 \text{ and } |x_j| > |x_k| \; \forall k \neq j\}$$

for $j = 1, \ldots, d$. Moreover, we denote by $\overline{Q_j^{\sim}}$ the corresponding closure of the cone. Finally, we define the sum of weights of all the points in the interior of a cone by

$$W(X_j^{\sim}) := \sum_{P_i \in Q_j^{\sim}} w_i.$$

for $\sim \in \{\leq, \geq\}$. For an illustration of the definition of the cones the reader is referred to Fig. 1.

The above definitions and Theorem 3.2 immediately imply the following corollary for $d = 2$. However, we will see later that this is also true for $d \geq 3$.

Corollary 3.4. *Given n points $P_i = (x_1^i, x_2^i)$ $(i = 1, \dots, n)$ with vertex weights $w_i \geq 0$ such that $|x_1^i| \neq |x_2^i|$ for all $i = 1, \dots, n$. Then, $P^* = (0, 0)$ is an optimal solution of the 1-median problem in $\mathbb{R}^2$ with respect to the Chebyshev-norm if and only if*

$$W(X_1^{\geq}) = W(X_1^{\leq}) \quad \text{and} \quad W(X_2^{\geq}) = W(X_2^{\leq}) \tag{11}$$

hold.

Proof. Due to the fact that $|x_1^i| \neq |x_2^i|$ holds we know by definition that $W(X_1 > X_2) = W(X_1 \geq X_2)$ and $W(X_1 < X_2) = W(X_1 \leq X_2)$ is satisfied. Then, (6) and (7) imply $W(X_1 > X_2) = W(X_1 < X_2)$.

Moreover, we have by definition that $W(X_1 > X_2) = W(X_1^{\geq}) + W(X_2^{\leq})$ is satisfied and $W(X_1 < X_2) = W(X_1^{\leq}) + W(X_2^{\geq})$ holds. Thus, we can conclude that

$$W(X_1^{\geq}) + W(X_2^{\leq}) = W(X_1^{\leq}) + W(X_2^{\geq}).$$

Using the same arguments

$$W(X_1^{\leq}) + W(X_2^{\leq}) = W(X_1^{\geq}) + W(X_2^{\geq})$$

can also be obtained. If we solve the system of those two linear equations it can be seen that (11) is equivalent to (6)–(9) and the result follows. □

In the data from Example 3.3 the points P_2 and P_4 do not satisfy the condition of the above corollary, because these two points are not in the interior of the cones. Thus, we cannot make use of condition (11) directly in order to prove that the origin is an optimal solution. However, note that we can assign the weights of P_2 and P_4 to the cones that contain these two points in its closure by splitting the weights in the following way:

1. P_4 is contained in the closures of $Q_1^{\geq}$ and $Q_2^{\leq}$. Let us assign one weight unit to each of these two cones. Thereby, we increase $W(X_1^{\geq})$ and $W(X_2^{\leq})$ by one.
2. P_2 is contained in the closures of the cones $Q_1^{\geq}$ and $Q_2^{\geq}$. Here the total weight, i.e., one unit of weight, is assigned to $Q_1^{\geq}$, which increases $W(X_1^{\geq})$ by one, whereas $W(X_2^{\geq})$ remains unchanged.

After this splitting of the weights condition (11) of Corollary 3.4 is satisfied, because the weight in each cone equals 3. Note that we are only allowed to split a weight of a point that is contained in the closure of at least two cones. Moreover, we are only allowed to distribute the weight to these cones.

In the following sections, we show that the above example is not a coincidence. In fact, we will prove that the origin is an optimal solution of the 1-median problem with respect to the Chebyshev-norm in the d-dimensional space if and only if there is a weight splitting such that condition (11) is satisfied after the splitting.

4. The optimality criterion

In this section, we state the optimality criterion discussed in the previous section more precisely. Before this can be done we introduce the 2-balanced flow problem on general graphs. It turns out later that this problem defined on special bipartite graphs is closely related to the dual problem of the LP given in (2)–(5).

4.1. The 2-balanced flow problem

Let $G = (V, E)$ be a directed graph, $s \in V$ a source and $T = \{t_1, \dots, t_{2l}\} \subset V \setminus \{s\}$ a set of an even number of sinks. Furthermore, we assume without loss of generality that each sink t_i $(i = 1, \dots, 2l)$ can be reached from the source and that there do not exist any edges that leave a sink or enter the source. Moreover, let $u(v, w)$ or $u(e)$ be the non-negative capacity associated with an edge $e = (v, w)$ for all $e \in E$. A flow on G is a non-negative real-valued function $f : E \to \mathbb{R}_{\geq 0}$ on the edges which satisfies the capacity constraints and the flow conservation constraints, i.e., $0 \leq f(e) \leq u(e)$ for all $e \in E$ and

$$\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = 0 \quad \forall v \in V \setminus (\{s\} \cup T),$$

where $\delta^+(v)$ and $\delta^-(v)$ denote the set of all edges that leave vertex v, respectively enter vertex v. For a given flow f we define the excess of a sink $t_i \in T$ with respect to f as

$$\mathrm{ex}_f(t_i) := \sum_{e \in \delta^-(t_i)} f(e),$$

i.e., the amount of flow entering the sink. As in the classic flow problem the value of a flow is defined as

$$v(f) := \sum_{e \in \delta^+(s)} f(e).$$

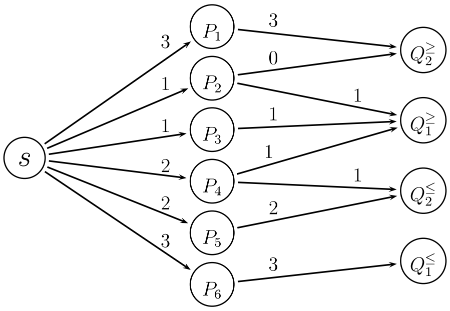

Fig. 2. On the edges a 2-balanced flow is given which shows that the origin is an optimal solution for the 1-median in Example 3.3.

Moreover, we define the imbalance of a flow f as

$$\sum_{i=1}^{l} |\mathrm{ex}_f(t_{2i}) - \mathrm{ex}_f(t_{2i-1})|.$$

and call f 2-balanced if the imbalance is 0. In the following, two sinks t_{2i} and t_{2i-i} $(i = 1, \ldots, l)$ will be called a pair of sinks. Thus the flow excess of a 2-balanced flow has to be the same for paired sinks. The task of the 2-balanced flow problem is to find a 2-balanced flow with maximum flow value. An instance of the 2-balanced flow problem with an optimal flow is shown in Fig. 2.

4.2. The dual problem

Let us consider the dual linear programming problem of (2)–(5). We introduce non-negative dual variables $u_{i,j}$ for the constraints (3) and non-negative dual variables $v_{i,j}$ for the constraints (4). Then, the dual problem reads as follows:

$$\max \sum_{i=1}^{n}\sum_{j=1}^{d} x_j^i \left(u_{i,j} - v_{i,j}\right) \tag{12}$$

$$\text{s.t.} \quad \sum_{j=1}^{d} \left(u_{i,j} + v_{i,j}\right) = w_i \quad i = 1, \ldots, n \tag{13}$$

$$\sum_{i=1}^{n} u_{i,j} - \sum_{i=1}^{n} v_{i,j} = 0 \quad j = 1, \ldots, d \tag{14}$$

$$u_{i,j} \geq 0,\ v_{i,j} \geq 0 \quad i = 1, \ldots, n,\ j = 1, \ldots, d. \tag{15}$$

In the following it is shown that the dual problem derived above is closely related to the 2-balanced flow problem. Consider the following instance of the 2-balanced flow problem: The graph $G = (V_1 \cup V_2, E)$ is a bipartite graph, where the set V_1 consists of n vertices, one for each point P_i $(i = 1, \ldots, n)$ of the 1-median problem. The set V_2 has exactly $2d$ vertices representing the cones $Q_j^{\geq}$ and $Q_j^{\leq}$, where $Q_j^{\geq}$ and $Q_j^{\leq}$ form a pair $(j = 1, \ldots, d)$ in the 2-balanced flow problem. Furthermore, we add a source s and edges (s, P_i) for all $i = 1, \ldots, n$ with an upper capacity of $u(s, P_i) = w_i$. Note that there is no lower bound on the flow, i.e., $0 \leq f(e) \leq w_i$ has to be satisfied for those edges. Finally, there is an edge from a point P_i to a cone $Q_j^{\sim}$ if and only if $P_i \in \overline{Q_j^{\sim}}$ $(\sim \in \{\leq, \geq\})$ and all these edges have infinite capacity. For a given instance of the location problem with the points $P_1, \ldots, P_n$ and a weight vector w the instance of the 2-balanced flow problem introduced above will be denoted by $I(P_1, \ldots, P_n, w)$.

The next theorem states the relation between the 2-balanced flow problem and the location problem.

Theorem 4.1. *Suppose we are given n points $P_1, \ldots, P_n \in \mathbb{R}^d$ with non-negative weights $w_i \geq 0$ for all $i = 1, \ldots, n$. Then, the origin $P^* = (0, \ldots, 0)$ is an optimal solution of the 1-median problem if and only if there exists a 2-balanced flow f with*

$$v(f) = \sum_{i=1}^{n} w_i \tag{16}$$

in the instance $I(P_1, \ldots, P_n, w)$.

Especially, we have $f(s, P_i) = w_i$ for all $i = 1, \ldots, n$, i.e., all the arcs leaving the source s are saturated.

Proof. Let us assume that the origin is an optimal solution of the primal problem, i.e., $y_j^* = 0$ for all $j = 1, \ldots, d$ and

$$z_i^* = \max_{j=1,\ldots,d} \{x_j^i, -x_j^i\} \quad \text{for all } i = 1, \ldots, n.$$

In order to obtain an optimal solution of the dual problem we know from the Complementary Slackness Theorem that if $z_i^* > x_j^i$ then $u_{i,j}^* = 0$ and if $z_i^* > -x_j^i$ then $v_{i,j}^* = 0$. Let us define the two sets

$$U = \{u_{i,j} : z_i^* = x_j^i\} \quad \text{and} \quad V = \{v_{i,j} : z_i^* = -x_j^i\}.$$

Note that all the constraints in the dual problem are equations. Thus, we can conclude that a feasible solution of the dual problem is optimal if and only if the only variables that are positive are in the sets U and V. Due to the fact that the optimal objective function value of the primal problem (2)–(5) is bounded from below by 0, we can conclude by the Strong Duality Theorem that the dual problem (12)–(15) also has an optimal solution. Thus, there exists an optimal solution of the dual problem satisfying

$$u_{i,j} = 0 \quad \text{and} \quad v_{i,j} = 0 \quad \forall u_{i,j} \notin U,\ v_{i,j} \notin V. \tag{17}$$

This solution is a 2-balanced flow in the instance $I(P_1, \ldots, P_n, w)$, where the flow on the edges is given by the variables $u_{i,j}$ and $v_{i,j}$. More precisely, $u_{i,j}$ gives the flow on the edge $(P_i, Q_j^{\geq})$, $v_{i,j}$ gives the flow on the edge $(P_i, Q_j^{\leq})$ and

$$f(s, P_i) = \sum_{j=1}^{d} \left(u_{i,j} + v_{i,j}\right).$$

Using this interpretation of the dual variables equation (13) guarantees that the flow value equals the sum of the weights of the points. Moreover, constraint (14) provides a 2-balanced flow, because the excess in $Q_j^{\geq}$ and $Q_j^{\leq}$ is the same for all $j = 1, \ldots, d$. Finally, note that due to the definition of the sets U and V the constraints given in (17) mean that $u_{i,j}$ and $v_{i,j}$ are only positive if the corresponding edge is in the instance $I(P_1, \ldots, P_n, w)$.

On the other hand, assume that the origin is not an optimal solution. Then, we can again argue by the Complementary Slackness Theorem that the dual problem (12)–(15) together with the constraints (17) does not have a feasible solution. This implies immediately that the instance $I(P_1, \ldots, P_n, w)$ does not permit a 2-balanced flow f satisfying Eq. (16). □

Example 4.2. We can now show using Theorem 4.1 that the origin is an optimal solution of the 1-median problem in Example 3.3. In Fig. 2 the corresponding instance $I(P_1, \ldots, P_n, w)$ is shown and a 2-balanced flow whose flow value equals the sum of the weights of the points is given.

5. A combinatorial algorithm for the inverse problem

In this section, we give an algorithm that solves the inverse location problem. The idea is mainly based on the optimality criterion of the previous section. In fact, we have to change the weights of the problem at minimum cost such that the corresponding 2-balanced flow problem permits an optimal flow f whose flow value equals the sum of the new weights. At the beginning of this section we show that there is always an optimal solution of the inverse problem such that the original weights are not increased. Afterwards, we introduce the fractional b-matching problem and show that there is a very strong relationship between the fractional b-matching problem and instances of the 2-balanced flow problem that are considered in this paper. Finally, we give a combinatorial algorithm for the inverse location problem.

5.1. The inverse problem

Before we propose a solution method for the inverse location problem we show that there always exists an optimal solution in which no weight is increased.

Lemma 5.1. *There exists an optimal solution w^* of the inverse location problem such that $w_i \geq w_i^*$ holds for all $i = 1, \ldots, n$.*

Proof. Let us define for a feasible solution $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n)$ of the inverse location problem the quantity

$$D(\bar{w}) := \sum_{i=1}^{n} \max\left((\bar{w}_i - w_i), 0\right).$$

Note that $D(\bar{w}) \geq 0$ holds for all feasible solutions $\bar{w}$ and $D(\bar{w}) = 0$ is satisfied if and only if $w_i \geq \bar{w}_i$ for all $i = 1 \ldots, n$. Thus, it remains to show that there exists an optimal solution $w^* = (w_1^*, \ldots, w_n^*)$ with $D(w^*) = 0$. Suppose that w^* is an optimal solution with $0 < D(w^*) \leq D(\bar{w})$ for all optimal solutions $\bar{w}$. Moreover, we assume without loss of generality that $w_1^* > w_1$ holds.

Due to the fact that w^* is an optimal solution we know from Theorem 4.1 that there exists a 2-balanced flow f^* in the instance $I(P_1, \ldots, P_n, w^*)$ with $f^*(s, P_i) = w_i^*$ for all $i = 1, \ldots, n$. In particular, we have $f^*(s, P_1) > 0$. Therefore there is at least one pair of sinks, denoted by $\hat{Q}$ and $\bar{Q}$, and a vertex P_j for some $j = 1, \dots, n$ such that the flow values $f^*(P_1, \bar{Q}) =: \varepsilon_1$, $f^*(s, P_j) = w_j^*$ and $f^*(P_j, \hat{Q}) =: \varepsilon_2$ are all positive. Note that for the remaining part of the proof it is not important which pair is taken if there is more than one such pair.

Let us distinguish two different cases:

- $j \neq 1$

  In this case, we construct a new 2-balanced flow f' by setting

  $$\varepsilon := \min\left\{\varepsilon_1, \varepsilon_2, w_1^* - w_1, w_j^*\right\}$$

  and by defining

  $$f'(u, v) := \begin{cases} f^*(u, v) - \varepsilon & \text{if } (u, v) \in \{(s, P_1), (P_1, \bar{Q}), (s, P_j), (P_j, \hat{Q})\} \\ f^*(u, v) & \text{otherwise} \end{cases}.$$

  This new 2-balanced flow gives also a new solution $w' = (w_1', \dots, w_n')$ for the inverse location problem with $w_i' = w_i^*$ for all $i \neq 1, j$ and $w_i' = w_i^* - \varepsilon$ for $i = 1, j$. Thus, we can conclude immediately that $D(w^*) > D(w')$.

  Let us now show that w' is also an optimal solution by considering the cost difference

  $$\begin{aligned} \sum_{i=1}^{n} |w_i^* - w_i| - \sum_{i=1}^{n} |w_i' - w_i| &= (w_1^* - w_1) - (w_1' - w_1) + |w_j^* - w_j| - |w_j' - w_j| \\ &= (w_1^* - w_1) - (w_1^* - \varepsilon - w_1) + |w_j^* - w_j| - |w_j^* - \varepsilon - w_j| \\ &= \varepsilon + |w_j^* - w_j| - |w_j^* - \varepsilon - w_j| \geq 0. \end{aligned}$$

  These facts lead to a contradiction to the choice of w^*.

- $j = 1$

  This situation can be handled similarly to the previous case. We define

  $$\varepsilon := \min\left\{\varepsilon_1, \varepsilon_2, \frac{w_1^* - w_1}{2}\right\}$$

  and obtain a new 2-balanced flow f' by

  $$f'(u, v) := \begin{cases} f^*(u, v) - \varepsilon & \text{if } (u, v) \in \{(P_1, \bar{Q}), (P_j, \hat{Q})\} \\ f^*(u, v) - 2\varepsilon & \text{if } (u, v) = (s, P_1) \\ f^*(u, v) & \text{otherwise} \end{cases}.$$

  This gives again a new feasible solution w' for the inverse location problem with $w_i' = w_i^*$ for all $i = 2, \dots, n$ and $w_1' = w_1^* - 2\varepsilon$. Comparing the costs of the different solutions leads to

  $$\sum_{i=1}^{n} |w_i^* - w_i| - \sum_{i=1}^{n} |w_i' - w_i| = (w_1^* - w_1) - (w_1' - w_1) = 2\varepsilon > 0$$

  which is a contradiction to the optimality of w^*. □

Using the above arguments it is clear that we can solve the inverse location problem using Algorithm 1.

Algorithm 1 Algorithm for the inverse location problem

1: Construct the instance $I(P_1, \dots, P_n, w)$
2: Find a maximum 2-balanced flow f^* in $I(P_1, \dots, P_n, w)$
3: The optimal solution of the inverse problem is given by $w_i^* = f^*(s, P_i)$

Note that we can also conclude that the optimal objective function value of the inverse problem is given by

$$\sum_{i=1}^{n} \left(w_i - w_i^*\right) = \sum_{i=1}^{n} \left(w_i - f^*(s, P_i)\right) = \left(\sum_{i=1}^{n} w_i\right) - v(f^*).$$

The main step of Algorithm 1 is obviously the computation of a maximum 2-balanced flow in Step 2. An algorithm for this problem and its complexity is discussed in the next subsections.

5.2. The fractional b-matching problem

Before we define the fractional b-matching problem we describe the b-matching problem. Given is an undirected graph $G = (V, E)$ with numbers $b : V \to \mathbb{R}_{\geq 0}$ and a profit function $p : E \to \mathbb{R}_{\geq 0}$. The task is to find $\alpha(i, j) \in \mathbb{N}$ for all $(i, j) \in E$ such that

$$v'(\alpha) := \sum_{(i,j)\in E} p(i, j)\alpha(i, j)$$

is maximized and

$$\sum_{(i,j)\in E} \alpha(i, j) \leq b(i) \quad \forall i \in V. \tag{18}$$

Note that if $b(v) = 1$ for all $v \in V$ and $p(i, j) = 1$ for all $(i, j) \in E$ the b-matching problem is the classical maximum matching problem in G. If we relax $\alpha(i, j) \in \mathbb{N}$ to $\alpha(i, j) \in \mathbb{R}_{\geq 0}$ the problem is called fractional b-matching problem. Furthermore, a (fractional) b-matching $\alpha(i, j)$ is called perfect if

$$\sum_{(i,j)\in E} \alpha(i, j) = b(i) \quad \forall i \in V.$$

It is well known that the fractional and integral b-matching problem can be solved in polynomial time. The fractional version can even be rewritten as a min-cost-flow problem in a bipartite graph (see e.g., Anstee [13]). For more details about the b-matching problem the reader is referred to Schrijver [14].

5.3. An algorithm for an instance $I(P_1, \ldots, P_n, w)$

In this subsection, it is shown how an instance $I(P_1, \ldots, P_n, w)$ of the 2-balanced flow problem can be transformed to a fractional b-matching problem. At the beginning we make the assumption that no given point of the location problem is at located at the origin, i.e., there is no point contained in $\overline{Q_j^{\geq}} \cap \overline{Q_j^{\leq}}$ for all $j = 1, \ldots, d$. This assumption avoids some technical problems and makes the explanation of the main ideas easier. Later on, we will discuss the general case where this assumption is dropped.

5.3.1. No given point is located at the origin

Given an instance $I(P_1, \ldots, P_n, w)$ and the corresponding network $G = (V, E)$ we define an undirected graph $G' = (V', E')$ with n vertices, one for each point P_i, i.e., $|V'| = n$. Moreover, $(P_i, P_k) \in E'$ if and only if there is an edge $(P_i, Q_j^{\geq}) \in E$ and an edge $(P_k, Q_j^{\leq}) \in E$ for some $j = 1, \ldots, d$ in the 2-balanced flow instance. Note that the graph G' does not have any loops. Finally, we set $b(P_i) = w_i$ and $p(e) = 1$ for all $e \in E'$.

The following algorithms show that in order to find a maximum 2-balanced flow it suffices to find a maximum fractional b-matching in $G' = (V', E')$ and vice versa.

Algorithm 2 has a given fractional matching $\alpha(i, j)$ in G' as input and constructs a 2-balanced flow in G. The basic idea is to start with the flow $f(e) = 0$. Then all edges $(P_i, P_j) \in E'$ with $\alpha(P_i, P_j) > 0$ are considered and the flow is updated in such a way that it is still 2-balanced and has a flow value which is increased by $2\alpha(P_i, P_j)$.

Algorithm 2 Construction of a 2-balanced flow f in $G = (V, E)$ given a fractional b-matching α in $G' = (V', E')$ with $2v'(\alpha) = v(f)$

```
Set f(e) = 0 for all e ∈ E
while there exists an edge with α(P_i, P_k) > 0 do
    {Assume w.l.o.g. P_i ∈ cl(Q_j^≥), P_k ∈ cl(Q_j^≤)}
    Increase the flow on (s, P_i), (s, P_k), (P_k, Q_j^≤) and (P_i, Q_j^≥) by α(P_i, P_k)
    Set α(P_i, P_k) = 0
end while
```

Lemma 5.2. *Algorithm 2 constructs for any fractional b-matching $\alpha(P_i, P_k)$ in G' a 2-balanced flow f in G with*

$$2v'(\alpha) = v(f). \tag{19}$$

Proof. At the beginning of the algorithm we have a 2-balanced flow with $v(f) = 0$. After each iteration of the WHILE-loop the updated flow is still 2-balanced because the excess of $Q_j^{\leq}$ and $Q_j^{\geq}$, two sinks that form a pair, are increased by the same amount. Moreover, the flow value increases by $2\alpha(P_i, P_k)$. Thus, we have

$$f(s, P_i) = \sum_{(P_i,P_k)\in E'} \alpha(P_i, P_k) \leq b(P_i) = w_i$$

at the end of the algorithm which means that the 2-balanced flow also satisfies the capacity constraints. Hence, the output of the algorithm is a 2-balanced flow f in G with

$$2 \sum_{(P_i,P_k)\in E'} \alpha(P_i, P_k) = 2v'(\alpha) = v(f). \quad \square$$

On the other hand, using Algorithm 3 we can also compute a fractional b-matching in G' if a 2-balanced flow f in G is given. Here, we start with the fractional b-matching $\alpha(P_i, P_k) = 0$ for all edges in G'. While decreasing the flow value of the 2-balanced flow in G we update the fractional b-matching in such a way that its objective function value increases.

Algorithm 3 Construction of a fractional b-matching α in $G' = (V', E')$ given a 2-balanced flow f in $G = (V, E)$ with $2v'(\alpha) = v(f)$

1: Set $\alpha(P_i, P_j) = 0$ for all $e \in E'$
2: **while** there exist two edges $(P_i, Q_j^{\geq})$ and $(P_k, Q_j^{\leq})$ with positive flow **do**
3: Let $\varepsilon := \min\{f(P_i, Q_j^{\geq}), f(P_k, Q_j^{\leq})\}$
4: Increase $\alpha(P_i, P_k)$ by ε
5: Decrease the flow on the edges $(P_i, Q_j^{\geq})$, $(P_k, Q_j^{\leq})$, (s, P_i) and (s, P_k) by ε
6: **end while**

Lemma 5.3. *Algorithm* 3 *constructs for any* 2*-balanced flow f in G a fractional b-matching $\alpha(P_i, P_k)$ in G' such that equation (19) holds.*

Proof. At the beginning of the algorithm we have a fractional b-matching with $v'(\alpha) = 0$. In each iteration we increase the objective function value of the fractional b-matching by ε and decrease the flow value by 2ε. Moreover, the updated flow remains 2-balanced. Thus, we obtain an objective function value of $\frac{1}{2}v(f)$ for the fractional b-matching in G' after the algorithm terminates. Moreover,

$$b(P_i) = w_i \geq f(s, P_i) = \sum_{(P_i,P_k)\in E'} \alpha(P_i, P_k)$$

holds and it can be concluded that α is indeed a feasible fractional b-matching in G'. $\square$

The above discussion leads to the following corollary.

Corollary 5.4. *Let an instance of the* 1*-median problem be given where no point is located in the origin. Then, the origin is a* 1*-median if and only if there exists a perfect fractional b-matching in $G' = (V', E')$.*

Moreover, we want to point out that a maximum fractional matching α in G' can directly be used to compute the optimal new weights w_i^* of the inverse 1-median problem by

$$w_i^* = w_i - \sum_{(P_i,P_k)\in E'} \alpha(P_i, P_k)$$

and α is indeed a perfect matching with respect to the new weights.

In Fig. 3 we show a perfect fractional b-matching for the inverse location problem with the data given in Example 3.3.

5.3.2. A given point is located at the origin

At the end of this paper, we discuss the case where the origin is also a point of the original 1-median problem. Let us assume without loss of generality that $P_1 = P_0$. In this special situation we have an edge from P_1 to $Q_j^{\geq}$ and to $Q_j^{\leq}$ for all $j = 1, \dots, d$ in the 2-balanced flow problem $I(P_1, \dots, P_n, w)$. In order to solve this maximum 2-balanced flow problem we construct again a fractional b-matching problem.

We start by deleting P_1 from the location problem and construct the 2-balanced flow instance $I(P_2, \dots, P_n, \bar{w})$ where $\bar{w} = (w_2, \dots, w_n)$. For this instance we can now construct a fractional b-matching problem as described in the previous subsection. Note that the graph in the fractional b-matching problem has now only $n - 1$ vertices and $b(P_i) = w_i$ for $i = 2, \dots, n$. In the following we denote this graph by $\bar{G} = (\bar{V}, \bar{E})$. Using this notation we can now state the main lemma.

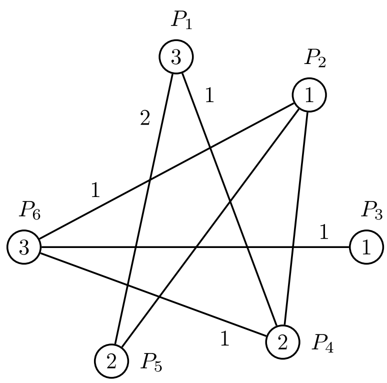

Fig. 3. The fractional b-matching shown in the graph corresponds to the maximum 2-balanced flow in Fig. 2.

Lemma 5.5. *Let $\bar{\alpha}(P_i, P_k)$ be an optimal fractional b-matching in $\bar{G} = (\bar{V}, \bar{E})$ with*

$$\sum_{i\in\bar{V}}\left(b(P_i) - \sum_{(P_i,P_k)\in\bar{E}}\alpha(P_i, P_k)\right) \le w_1. \tag{20}$$

Then, P_1 is a 1-median of the location problem.

Proof. Let us assume that $\bar{\alpha}(P_i, P_k)$ is an optimal fractional b-matching in $\bar{G}$ which satisfies (20). Using Algorithm 3 we can obtain a 2-balanced flow $\bar{f}$ in $I(P_2, \dots, P_n, \bar{w})$ with $v(\bar{f}) = 2v'(\bar{\alpha})$.

In order to extend the network given in $I(P_2, \dots, P_n, \bar{w})$ to the network in $I(P_1, \dots, P_n, w)$ we add the P_1 and the edges (s, P_1), $(P_1, Q_j^{\le})$ and $(P_1, Q_j^{\ge})$ for all $j = 1, \dots, d$ with $u(s, P_1) = w_1$. Furthermore, we define a 2-balanced flow f in this extended network by $\bar{f}(e) = 0$ for the new edges and $f(e) = \bar{f}(e)$ for the other edges. Thus it follows immediately that the flow value of both flows are the same. More precisely, we have

$$\sum_{i\in\bar{V}}\sum_{(P_i,P_k)\in\bar{E}}\alpha(P_i, P_k) = 2v'(\bar{\alpha}) = \sum_{i=2}^{n} f(s, P_i)$$

because each edge value $\alpha(P_i, P_k)$ is counted twice. As a consequence we get the following bound

$$\sum_{i=2}^{n}(w_i - f(s, P_i)) \le w_1. \tag{21}$$

In order to finish the proof we construct a new 2-balanced flow with flow value $\sum_{i=1}^{n} w_i$ by starting with the flow f. For $i = 2, \dots, n$ with $f(s, P_i) < w_i$ we do the following: Let $\varepsilon = w_i - f(s, P_i)$ and denote by $\hat{Q}$ a vertex for which an edge $(P_i, \hat{Q})$ exists. Finally, we assume that $\hat{Q}$ and $\bar{Q}$ form a pair of sinks. We increase the flow on the edges (s, P_i), $(P_i, \hat{Q})$, (s, P_1) and $(P_1, \bar{Q})$ by ε. It follows from the definition that the new flow is again 2-balanced.

Note that due to inequality (21) this procedure can indeed be done for all $i \ne 1$ without exceeding the capacity of the edge (s, P_1). Thus, we obtain a 2-balanced flow which saturates the edges (s, P_i) for all $i = 2, \dots, n$.

Finally, we increase the flow on (s, P_1) until this edge is saturated and increase the flow on each of the edges $(P_1, Q_1^{\le})$ and $(P_1, Q_1^{\ge})$ by half of this amount. After this we have a 2-balanced flow in $I(P_1 \dots, P_n, w)$ with a flow value that equals the sum of all the weights of the original location problem. This proves that the origin is an optimal solution. □

From the proof of this lemma it should be clear that the construction of the 2-balanced flow can also be reversed, i.e., given a 2-balanced flow in $I(P_1, \dots, P_n, w)$ that saturates all the edges leaving s, one can obtain a fractional b-matching in $G' = (V', E')$ satisfying (20).

Concluding it can be said that in order to solve the inverse 1-median problem it suffices to solve a fractional b-matching problem. Due to the fact that the fractional b-matching problem can be transformed to a Hitchcock Transportation Problem which can be solved in $\mathcal{O}(n^3 \log n)$ time (see [13,14]), we have the following theorem.

Theorem 5.6. *The inverse 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm can be solved in $\mathcal{O}(n^3 \log n)$ time.*

6. Conclusion

This paper gives the first optimality criterion for the inverse 1-median problem in $\mathbb{R}^d$ with the Chebyshev-norm in arbitrary dimensions. Using this criterion the inverse problem is reduced to a maximum 2-balanced flow problem. The maximum 2-balanced flow problems in this paper have a very special structure. In fact, the underlying graph is bipartite with an additional source and only the edges leaving that source have finite capacity. However, to the best of our knowledge there is still no combinatorial algorithm known for the general maximum 2-balanced flow problem. This seems to be a challenging task for further research. It might also be interesting to consider the problem where we want to find a flow such that the flow value should be the same in more than two sinks.

Moreover, it might be interesting to consider the asymmetric ℓ_1-norm as objective function in the inverse problem, i.e., the cost for changing the weights depend on the vertices. In this more general situation it is not true that there always exists an optimal solution of the problem where all the weights are decreased. Another problem that could be addressed is the consideration of upper and lower bounds on the changes of the weights that are allowed.

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